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Two algorithms for symmetric groups

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Abstract. We describe two algorithms, one for the 'generation' of the elements of a certain class of subgroups of the symmetric group and the other for the 'generation' of the coset representatives of these subgroups in the symmetric group. Later we discuss the relevance of these algorithms in the enumeration of distinct and connected diagrams in many-body perturbation theory.

1. Introduction

Recently, in a series of papers, Rosensteel *et al* (Rosensteel *et al* 1975, Ihrig *et al* 1976) have analysed many-body perturbation-theoretic diagrams, using the properties of symmetric groups, and have characterised the topologically distinct and connected diagrams, in terms of equivalence classes with respect to the subgroup consisting of the elements of the centraliser of the permutation which labels the interaction lines in a natural order. In implementing these ideas, we need detailed information not only on the subgroup but also on its coset representatives in the symmetric group. To this end we can make use of some theorems in group theory (Kerber 1971). However, the aim of this paper is to develop simple algorithms which require familiarity with expansions of a permanent. In our opinion, this way of generating the coset representatives seems to be new and more direct than the existing procedures, especially with regard to subgroups of symmetric groups. As is well known, determinants and permanents are intimately related to the representations of symmetric groups.

In § 2 we outline an algorithm for generating the centraliser of the elements of the symmetric group through prescriptions on the expansion of a given permanent. In § 3 we describe a similar algorithm to generate the coset representatives of the subgroups of the symmetric group generated by the first algorithm. Section 4 gives an application of the algorithms to many-body perturbation theory, in particular with regard to writing down all the connected and topologically distinct many-body diagrams. In § 5 we compare our algorithms with the procedure outlined in the paper of Ihrig *et al* (1976).

2. Algorithm for generating the centralisers of elements in the symmetric group

In what follows, we adopt the notation of Rosensteel *et al* (1975). Consider any element σ belonging to the symmetric group S_{2n+1} acting on $(2n+1)$ symbols $0, 1, \ldots, 2n$. The centraliser of σ in S_{2n+1}, denoted by $c(\sigma)$, is the subgroup defined by

$$c(\sigma) = \{p \in S_{2n+1} | p\sigma = \sigma p\}. \tag{2.1}$$

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An important property of the elements of $c(\sigma)$ is that they map cycles of σ into cycles of the same length (Rosensteel *et al* 1975). We start with the algorithm for the centraliser of the special element

$$\tau_0=(0)(12)(34)\ldots(2n-1\ 2n)\in S_{2n+1} \tag{2.2}$$

which consists only of 2-cycles, apart from the 1-cycle containing the symbol 0.

We write the $(2n+1)\times(2n+1)$ permanent in the Sylvester notation, first introduced in the context of quantum field theory by Caianiello (1973):

$$\begin{pmatrix}0 & 1 & 2 & 3 & 4 & \ldots & 2n-1 & 2n\\ 0 & 1 & 2 & 3 & 4 & \ldots & 2n-1 & 2n\end{pmatrix}=\begin{vmatrix}[00] & [01] & [02] & \ldots & [02n]\\ [10] & [11] & [12] & \ldots & [12n]\\ \vdots & & \ddots & & \vdots\\ [2n0] & [2n1] & [2n2] & \ldots & [2n2n]\end{vmatrix} \tag{2.3}$$

where the $2n+1$ symbols $0,1,2,\ldots,2n$ are divided into $n+1$ blocks. We refer to the above permanent as the generating permanent. The significance of the vertical bars will be evident from the following.

In the usual expansion of the permanent we obtain products of 2-tuples $[ab]$, $0\leqslant a, b\leqslant 2n+1$. The 2-tuple $[ab]$ means that a goes into b. We call all such 2-tuples $[ab]$ 'half-transpositions'. It is obvious that we have

$$(ab)=[ab][ba] \qquad (abc)=[ab][bc][ca]. \tag{2.4}$$

We now expand the permanent in the usual way, but we drop all the terms containing those 'half-transpositions' in which, if in the rth block containing the symbols a, $a+1$ in the upper row, a goes to b or $b+1$ occurring in the lower row of some sth block, but $a+1$ does not go into $b+1$ or b. Collect all the remaining terms. Every term is made up of a chain of 'half-transpositions' which we can join together to obtain the permutations belonging to $c(\tau_0)$.

In order to obtain $c(\sigma)$ for any σ in S_{2n+1} we proceed as follows. We keep in mind the crucial property that elements of S_{2n+1} map cycles of σ into cycles of the same length. Let σ be specified by its cycle decomposition, i.e. $\sigma=\Pi_{i=1}^{m}\sigma_i$ where σ_i denotes a cycle of length k_i. If l_j gives the number of σ_i with $k_i=j$, we have $\Sigma_j jl_j=(2n+1)$, $\Sigma_j l_j=m$.

We now write the $(2n+1)\times(2n+1)$ permanent by arranging rows in such a way that symbols belonging to cycles of the same length occur in a sequence. If we now partition this permanent by means of vertical bars, it will consist of blocks of size $jl_j\times jl_j$, which can be further partitioned into l_j smaller blocks of size $j\times j$ in such a way that these partitions reflect exactly the cycle structure of σ.

To obtain $c(\sigma)$ we write the $(2n+1)\times(2n+1)$ permanent as a product of factors which are permanents of size $jl_j\times jl_j$. Each factor consists of l_j permanents in which upper and lower rows will contain j symbols forming the j-cycles in σ. We now express each factor as a sum of $l_j!$ terms where every term will be a product of permanents of size $j\times j$. We have $l_j!$ terms because we permute the l_j sets of symbols in the lower rows without disturbing the l_j sets of j symbols in the upper rows which reflect exactly the cycle structure of σ. While permuting, each set is replaced by another set without changing the order of the symbols within any single set.

We now expand the $j\times j$ permanents and retain only those terms according to the following prescriptions.

(a) If the symbols in the upper and lower rows in a $j \times j$ permanent are the same, we write the terms of the cyclic group generated by the symbols as half transpositions, i.e.

$$\begin{pmatrix} p & q & r \\ p & q & r \end{pmatrix} \Rightarrow \{[pp][qq][rr], [pq][qr][rp], [pr][rq][qp]\}. \quad (2.5)$$

(b) When the symbols in the lower row are different from those in the upper row, we replace the second symbol in each of the above half transpositions (2.5) by the symbol corresponding to it in the lower row, i.e.

$$\begin{pmatrix} p & q & r \\ s & t & u \end{pmatrix} \Rightarrow \{[ps][qt][ru], [pt][qu][rs], [pu][rt][qs]\}. \quad (2.6)$$

With these prescriptions we can easily write down $c(\sigma)$, by joining the half transpositions in the products.

It can be seen that the order of $c(\sigma)$ is given by

$$^{\circ}c(\sigma) = \prod_j l_j! j^{l_j}. \quad (2.7)$$

3. Generating the coset representatives of the centralisers of the elements of S_{2n+1}

The method due to Todd and Coxeter (1936; Coxeter and Moser 1965) for the systematic enumeration of coset representatives is applicable to any group–subgroup pair, but it requires a prior knowledge of the group generators. In the following we describe a method for the systematic enumeration of the coset representatives of $c(\sigma)$ in S_{2n+1}. This method does not require any knowledge of the group generators. It is also different from the method given in Ihrig *et al* (1976).

For this purpose partition the $(2n+1) \times (2n+1)$ permanent as before. Expand the permanent in the standard way but with the following proviso.

(a) If confronted with a succession of complete blocks of equal lengths, expand only by one of the symbols occurring in the first complete block in that succession of blocks. Here a complete block means a block in which upper and lower rows are identical. Blocks which are not complete are called incomplete blocks. Drop all the other terms.

(b) While expanding with respect to symbols occurring in an incomplete block, we expand with respect to all the symbols in the lower row. We continue this expansion process until we are left with only 2×2 permanents, preceded by a chain of 'half-transpositions'. Finally, to obtain the coset representatives, expand these 2×2 permanents in the standard way, and piece together all the 'half-transpositions' to convert them into permutations.

However, for the special case of $\tau_0 = (0)(12)(34)\ldots(2n-1\ 2n)$, all the terms obtained by the above expansion are not coset representatives. As a rule one has to drop all the first terms obtained by expansion of those 2×2 determinants which have the structure of complete blocks. To make up for the loss of this number of coset representatives, one has to add the set of coset representatives of S_{2n} to the set obtained from S_{2n+1} by the above expansion. The former set consists of the coset representatives of $c(\tau_0)$, $\tau_0 = (12)(34)\ldots(2n-1\ 2n)$. We note here that in order to make these elements of S_{2n} become elements of S_{2n+1} one has to add a 1-cycle (0) to each coset representative of S_{2n}. Therefore, the method of writing a complete system of coset representatives of $c(\tau_0)$ is essentially a recursive procedure.

4. An application

We now consider an application of the above way of obtaining the coset representatives to the enumeration of many-body diagrams (Rosensteel *et al* 1975, Ihrig *et al* 1976). For this we consider only $c(\tau_0)$ and its coset representatives in S_{2n+1} corresponding to nth-order perturbation.

To start with, we notice that the coset representatives finally obtained by us are a collection of three different sets. These sets comprise the following coset representatives:

(i) those which come from the complete 2×2 blocks in the last step and some elements which essentially represent disconnected diagrams;

(ii) those which come from the incomplete 2×2 blocks in the last step;

(iii) those which come from the group S_{2n}.

We now observe that the set of coset representatives given by (ii) is exactly the one obtained by Caianiello's (1973) rule for generating all the connected and distinct graphs in the nth order of perturbation. We therefore conclude that all the connected and distinct graphs in any order n form a subset of the coset representatives of $c(\tau_0)$ in S_{2n+1}.

In the following we show how to describe all the connected and distinct many-body diagrams in any order n by making use of the permanent 'generating function' described previously. The method is essentially a recursive one. In order to write the nth-order connected and distinct diagrams, we have to know already all the connected and distinct graphs of all orders from zero to $n-1$. This can be seen as follows. Denote the set of coset representatives of $c_n(\tau_0)$ in S_{2n+1} by $S_{2n+1}/c_n(\tau_0)$. The cardinality of this set is ${}^{\circ}S_{2n+1}/{}^{\circ}c_n(\tau_0)$, the index of $c_n(\tau_0)$ in S_{2n+1}. We know that this set of coset representatives is the sum of three distinct sets given above,

$$S_{2n+1}/c_n(\tau_0)=S_{2n}/c_n(\tau_0)+N^c_{Gr}(n)+\text{terms from set (i)}. \tag{4.1}$$

The third term on the right-hand side refers to distinct but connected diagrams, i.e. each of the terms has a connected part of order less than n and a disconnected part which belongs to a symmetric group S_{2k} where $k<n$. Of course the number of symbols both in the connected and the disconnected parts put together is $2n+1$. These disconnected parts are all distinct coset representatives of S_{2k}, $k<n$. This is because the algorithms for the coset representatives are the same whether we have one block or n blocks. Moreover, because of the recursive nature of the algorithm, all the coset representatives for all S_{2k}, $k<n$, will be present as the disconnected parts of the coset representatives of S_{2n+1} for a given n. By the same reasoning, all the connected and distinct diagrams of order $<n$ will occur as the connected parts of the coset representatives of S_{2n+1} corresponding to the disconnected diagrams.

Therefore, from the above it is clear that we can write all the coset representatives of S_{2n+1} arising out of the third term on the right-hand side of the above equation as the following sum:

$$\sum_{m=1}^{n-1}(S_{2(n-m)}/c_{n-m}(\tau_0))N^c_{Gr}(m). \tag{4.2}$$

Therefore, we have for the set $N^c_{Gr}(n)$ the following equation:

$$S_{2n+1}/c_n(\tau_0)=N^c_{Gr}(n)+S_{2n}/c_n(\tau_0)+\sum_{m=1}^{n-1}(S_{2(n-m)}/c_{n-m}(\tau_0))N^c_{Gr}(m). \tag{4.3}$$

or

$$S_{2n+1}/c_n(\tau_0) = \sum_{m=0}^{n} (S_{2(n-m)}/c_{(n-m)}(\tau_0))N^c_{Gr}(m) \quad (4.4)$$

where we have made use of the fact that $S_0/c_0(\tau_0) = e$, the identity. As we see now, the above is a recursive relation for 'generating' the $N^c_{Gr}(n)$. We know that the order of the set on the left-hand side is equal to the order on the right-hand side. Hence we can write, using this fact,

$$°S_{2n+1}/°c_n(\tau_0) = \sum_{m=0}^{n} (°S_{2(n-m)}/°c_{n-m}(\tau_0))N^c_{Gr}(m). \quad (4.5)$$

In terms of pure numbers the above equation means

$$\frac{(2n+1)!}{2^n n!} = \sum_{m=0}^{n} \frac{[2(n-m)]!}{2^{n-m}(n-m)!} N^c_{Gr}(m) \quad (4.6)$$

where we substituted

$$°S_k = k!, \qquad °c_k(\tau_0) = 2^k k!. \quad (4.7)$$

Therefore, we conclude that the number of connected and distinct many-body diagrams in a given order of perturbative expansion of the many-body Hamiltonian can be evaluated in a recursive way, with the help of the above equation.

The last of the above equations can be written, in the double factorial notation, in the following manner:

$$(2n+1)!! = \sum_{m=0}^{n} [2(n-m)-1]!!N^c_{Gr}(m). \quad (4.8)$$

In this notation it is identically equal to the equation derived by Akyopan (1965) for the $N^c_{Gr}(m)$.

5. Conclusion

In conclusion, we wish to compare our algorithm with the procedures described by Ihrig *et al* (1976). In order to list all the distinct and connected graphs of a certain order, Ihrig *et al* (1976) first construct a transversal for $S_{2n+1}/c_n(\tau_0)$ and then proceed to list the elements representing the connected and distinct graphs by resorting to what they call 'method 2'. By this method they could enumerate the required diagrams only up to fourth order. We observe that our choice of the set of coset representatives (which they call the transversal of $S_{2n+1}/c_n(\tau_0)$) is a little more advantageous since it straightaway gives all the distinct and connected graphs up to any order n. In other words, Ihrig *et al* make the transversal a tool in writing all the connected and distinct graphs up to only fourth order, whereas for us the transversal itself provides the required diagrams of any order.

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